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On degree zero semistable bundles over an elliptic curve

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ABSTRACT

Motivated by the study of heterotic string compactifications on elliptically fibered Calabi-Yau manifolds, we present a procedure for testing semistability and identifying the decomposition type of degree zero holomorphic vector bundles over a nonsingular elliptic curve. The algorithm requires explicit knowledge of a basis of sections of an associated 'twisted bundle'.

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1 Introduction

In the study of a certain class of heterotic string compactifications one encounters the following:

Problem: *Given a smooth elliptic curve E and a degree zero holomorphic vector bundle V over E, find a practical algorithm to determine whether V is semistable. In this case, find a maximal decomposition of V in indecomposable subbundles.*

This question appeared in [1] in the course of an investigation of the relation between $(0,2)$ heterotic string compactifications and F-theory.

Since the most accessible bundle data are usually its global holomorphic sections, we will be interested in solving this problem by finding a characterization of semistability and of the decomposition type in terms of properties of a basis of sections of a certain bundle associated to V. Our main result is Theorem 2.2 in section 2.3.

Physical motivation: A $(0,2)$ heterotic compactification is characterized by a Calabi-Yau manifold Z and a stable holomorphic vector bundle V over Z [2]. If one is interested in models having a potential F-theory dual [3, 4, 5], one takes Z to be elliptically fibered and with a section. In this case, for a certain component of the moduli space, there exists an alternate description of stable vector bundles over Z in terms of pairs (Σ, L) where Σ is the spectral cover of V and L is a line bundle over Σ [5, 6, 7]. Such data are easier to manipulate then the abstract bundle data. On the other hand, there exists an accessible class of $(0,2)$ compactifications, namely those realized via $(0,2)$ linear sigma models [8, 9, 10, 11]. In this case, V is presented as the sheaf cohomology of a monad defined over Z, while Z itself is realized as a complete intersection in a toric variety [12]. This leads to the problem, studied in [1], of translating between these alternative presentations of V in the $(0,2)$ linear case. The main condition for V to admit a spectral cover description is that its restriction $V|_E$ to the generic elliptic fibre E of Z be semistable. In order to carry out the task of [1], one needed a method to test this condition for a given bundle V. This proves essential in organizing the wealth of models that can be built. An important point, which was tangentially mentioned in [1], is that the above condition often fails to hold, even for $(0,2)$ models which seem to be physically well-defined. One also finds a significant number of models for which the condition is satisfied but $V|_E$ does not fully decompose as a direct sum of line bundles. Discriminating between such cases can be achieved by the methods of the present paper. On the other hand, the method of [1] was justified only for the case when $V|_E$ is semistable and fully decomposable. Here we remedy this by providing a systematic discussion of the general situation.

Mathematical context: The main results we need date back to a classical paper of Atiyah [13] [1] . Fix a nonsingular elliptic curve E with a distinguished point p. Let $\mathcal{E}(r,0)$ be the set of (holomorphic equivalence classes of) *indecomposable* holomorphic vector bundles of rank r and degree zero over E. Any element $V \in \mathcal{E}(r,0)$ is of the form $V = L \otimes F_r$ with $L \in \mathrm{Pic}^0(E)$ a degree zero line bundle uniquely determined by

[1]Background material can be found for example in [14].

V and satisfying $L^r = \det V$. Here F_r is the unique element of $\mathcal{E}(r,0)$ with $h^0 \neq 0$. One has $h^0(F_r) = 1$. The bundles F_r can be defined inductively by $F_1 := O_E$ and by the fact that F_r is the unique nontrivial extension :

$$0 \longrightarrow F_{r-1} \longrightarrow F_r \longrightarrow O_E \longrightarrow 0 \tag{1}$$

of F_{r-1} by O_E. The Riemann-Roch theorem gives $h^1(F_r) = h^0(F_r) = 1$. It is known that F_r is semistable for all r.

For any holomorphic vector bundle of degree zero and rank r over E, consider a maximal decomposition as a direct sum of holomorphic subbundles:

$$V = \oplus_{j=1..k} V_j \tag{2}$$

If V is semistable, then we necessarily have $\deg V_j \leq 0$ for all $j = 1..k$ and since $0 = \deg V = \sum_{j=1..k} \deg V_j$, it follows that $\deg V_j = 0$ for all j. If $r_j := \mathrm{rank} V_j$, we thus have $V_j \in \mathcal{E}(r_j, 0)$ and $V_j = L_j \otimes F_{r_j}$, with $L_j \in \mathrm{Pic}^0(E)$. Thus :

$$V = \oplus_{j=1..k} L_j \otimes F_{r_j} \tag{3}$$

Note that $\sum_{j=1..k} r_j = r$.

Conversely, if such a decomposition of V exists, then, since all terms are semistable and of slope 0, a standard result (see [15, p17,Cor. 7]) assures us that V is semistable and of degree zero. The idea of our approach will be to use (3) in order to simultaneously check semistability and determine the maximal decomposition, thus avoiding the difficult problem of testing semistability independently.

The sequence of pairs $(r_j, L_j)(j = 1..k)$ will be called the *decomposition type* (or *splitting type*) of V. By using the distinguished point $p \in E$ to write $L_j \approx O(q_j - p)$ for some $q_j \in E$, we can identify this data with the sequence of pairs (r_j, q_j), modulo the choice of p. Obviously the splitting type determines V up to isomorphism.

Part of this information is encoded by what we will call the *spectral divisor* Σ_V of V, defined by :

$$\Sigma_V := r_1 q_1 + ... + r_k q_k \in \mathrm{Div}(E) \tag{4}$$

Note that some of the points q_j may coincide. If $q_1 = ... = q_{j_1} := Q_1$, ... , $q_{j_1+...+j_{l-1}+1} = .. = q_{j_1+...+j_l} := Q_l$ (with $j_1 + ... + j_l = k$), then $\Sigma_V = \rho_1 Q_1 + ... + \rho_l Q_l$ where

$$\rho_i = \sum_{j_1+...+j_{i-1}+1 \leq i \leq j_1+...+j_i} r_i. \tag{5}$$

In particular, Σ_V cannot discriminate between direct factors of the type $O(Q_1) \otimes (F_{r_1} \oplus ... \oplus F_{r_{j_1}})$ and factors of the type $O(Q_1) \otimes F_{r_1+...+r_{j_1}}$. In fact, it is easy to see [2] that

[2] Since the only stable bundles of slope zero over an elliptic curve are the degree zero line bundles, any Jordan-Holder (JH) filtration of V is by subbundles of consecutive dimension. The (isomorphism class of) the associated graded bundle $gr(V)$ is independent of the choice of the JH filtration. If V decomposes as above, the natural JH filtrations of F_{r_i} induce a JH filtration of V in the obvious way. The associated graded bundle is $gr(V) = O(Q_1 - p)^{\oplus \rho_1} \oplus ... \oplus O(Q_l - p)^{\oplus \rho_l}$. Therefore, Σ depends only on $gr(V)$, i.e. only on the S-equivalence class of V.

Σ_V depends only on the S-equivalence class of V. Two degree zero semistable vector bundles having the same spectral divisor need not have the same splitting type.

The explicit computation of Σ_V was the main task of [1]. In that paper, a solution of this problem was presented only for the 'fully split' case (this is rigorously formulated in Section 3). A by-product of the study we undertake here is a simple generalization of the method of [1] for determining the spectral divisor (see Corrolary 2.1 in section 2.3).

We will often consider the 'twisted' bundle $V' := V \otimes O(p)$, which has degree r and slope 1. If V is semistable, one has the following

Lemma 1.1 *Let V be a degree zero semistable vector bundle over E. Then $h^0(V') = \mathrm{rank} V$ and $h^1(V') = 0$.*

Proof: By (3), we have $h^0(V') = \sum_{j=1..k} h^0(O(q_j) \otimes F_{r_j})$. Since $O(q_j) \otimes F_{r_j}$ is indecomposable and of positive degree, a result of [13] shows that $h^0(O(q_j) \otimes F_{r_j}) = \mathrm{deg} O(q_j) \otimes F_{r_j} = r_j$ and the Riemann-Roch theorem gives $h^1(O(q_j) \otimes F_{r_j}) = 0$. This implies the conclusion. $\square$

As input data for the resolution of our problem we will assume explicit knowledge of a basis of sections of V'. This is typically easily computed, at least if V is presented as the sheaf cohomology of a monad. [3]

The plan of this paper is as follows. In section 2 we study semistability and the decomposition type for a degree zero holomorphic vector bundle V over E. We formulate necessary and sufficient conditions on a basis of sections of V' in order for V to be semistable; this will also indicate its decomposition type. In particular, we obtain a simple receipt for the spectral divisor. We also consider the spectral divisor in the monad case and propose a 'moduli problem'.

In section 3 we consider the fully decomposable ('fully split') case. We present a criterion for identifying fully decomposable and semistable vector bundles of degree zero over E, together with an algorithmic implementation. This is the main case considered in [1]. The novelty here is that the algorithm we give tests semistability of V (and at the same time determines its spectral divisor and its decomposition type, thus describing V completely in the language of [13]); in [1], the focus was on computing Σ_V and V was *assumed* to be semistable and fully decomposable in order to simplify the presentation. We also explain how one can analyze V by starting from more general twists. This is necessary in practice in cases when one cannot easily compute the sections of bundles over E twisted by $O(p)$. [4]. With the physics oriented reader in mind, the discussion

[3] Indeed, in that case one can consider the $O(p)$-twisted monad. The long exact cohomology sequence of the twisted monad will collapse due to the fact that $h^1(V') = 0$. This is one of the nice properties of V'.

[4] In the set-up of [1], one is interested in smooth elliptic curves realized as complete intersections in a toric variety $\mathbb{P}$. In this case, one can easily compute the sections of $V \otimes L_E$, for restrictions L_E of reflexive sheaves L over $\mathbb{P}$. If $O(p)$ is not such a restriction then the sections of $V \otimes O(p)$ are not easily accessible. For example, if E is realized as a cubic in $\mathbb{P}^2$, the line bundle $O_{\mathbb{P}^2}(1)$ over $\mathbb{P}^2$ restricts to a degree three line bundle $O(p_1 + p_2 + p_3)$ over E, and one can apply the methods of section 3 to the twisted bundle $V' := V \otimes O(p_1 + p_2 + p_3)$.

of section 3 is carried out by a direct approach and can be read independently of the rest of the paper; it is intended as a technical companion of [1].

Notation and terminology: If s is a regular section of a holomorphic vector bundle, then (s) denotes the zero divisor (divisor of zeroes) of s. $\mathrm{Div}(E)$ is the free abelian group of divisors on E. If $D \in \mathrm{Div}(E), D = \sum_{j=1..k} n_j p_j$, with $n_j \in \mathbb{Z}, p_j \in E$, then supp$D$ denotes the set $\{p_j | j = 1..k\}$. All vector bundles and their morphisms are holomorphic. For any vector space A, $\mathrm{Gr}^k(A)$ denotes the grassmanian of k-dimensional subspaces of A. If $S \in A$ is a subset, then $< S >$ denotes the linear span of S. For any holomorphic bundle R over E, $\mathrm{Gr}^k(R)$ denotes the set of rank k holomorphic subbundles of R. $\sim$ denotes linear equivalence of divisors and $\mathrm{Pic}(E)$ the Picard group of E. If r is an integer, then $\mathrm{Pic}^r(E)$ is the set of isomorphism classes of degree r line bundles over E; it is only a sub*set* of $\mathrm{Pic}(E)$, except for $r = 0$, when it is a subgroup. We say that a filtration $0 = K_0 \subset K_1 \subset ... \subset K_r = U$ of an r-dimenisonal vector space U is *nondegenerate* if $K_{i-1} \neq K_i$ for all $i = 1..r$. Then K_i have consecutive dimensions. For a holomorphic bundle V, we denote by $\mu(V) := \mathrm{deg}V/\mathrm{rank}V$ its slope (nomalized degree).

Intuitive idea The starting point for our analysis is the fact that the twisted bundles F'_r are given recursively as nontrivial extensions of $O(p)$ by itself. By Lemma 1.1, the associated cohomology sequences collapse and this provides a very good handle on the behaviour of F'_r. In terms of the local behaviour of sections, the difference between F'_r and the completely trivial extension $O(p)^{\oplus r}$ is manifest only at the point p. In both cases, the bundles admit a basis of r sections whose values are linearly independent at each point of E except p. At this point, the behaviour in the two cases is dramatically different. While in the completely decomposable case the values of all sections vanish simultaneously at p along linearly independent 'directions', in the case of F'_r only one of them vanishes, while the others have linearly independent values. In the latter case, however, the 'direction' of the first section approaches the space spanned by the values of the others as we approach p on E, and at the point p it lies in that space. The behaviour of the sections of V' can be obtained essentially by a 'linear superposition' from the behaviour of its indecomposable factors. Most of what follows consists in developing enough technology in order to make these ideas precise. This being understood, the physics-oriented reader may at first consider only the first part of subsection 2.1, the statements of Theorems 2.1 and 2.2 in section 2 and of Theorem 3.1 in section 3 and the associated algorithm.

2 General analysis

Let V be a degree zero holomorphic vector bundle over a smooth elliptic curve E. Fix a point $p \in E$ and define $V' := V \otimes O(p)$.

We present a criterion for deciding whether V is semistable and, in this case, for determining its splitting type. This criterion requires explicit knowledge of a basis of holomorphic sections of V'.

The plan of this section is as follows. In subsection 1 we discuss a notion of order of incidence of a holomorphic section on a subbundle. Since this discussion does not require assuming $\mathrm{deg}V = 0$, we will present it for a general holomorphic vector bundle over E. In subsection 2 we use these concepts to describe the sections of the bundles F'_r. In subsection 3 we give our characterization of degree zero semistable bundles.

2.1 Incidence order of holomorphic sections on subbundles

In this subsection let W be a rank r holomorphic vector bundle over E and let T be a rank r_0 holomorphic subbundle of W.

Any *nonzero* regular section of W defines a unique line subbundle L_s of W in the following way (see [13]). For each $t \in \mathrm{supp}(s)$, choose a local holomorphic coordinate z on E centered at t. Let ν_t be the degree of vanishing of s at t. Then $\exists \lim_{e\rightarrow p} z^{-\nu_t} s(e) := \hat{s}(t)$, where $\hat{s}(t) \in W_t - \{0\}$. We define $(L_s)_e :=< s(e) >$, for all $e \in E - \mathrm{supp}(s)$ and $(L_s)_t :=< \hat{s}(t) >$ for all $t \in \mathrm{supp}(s)$. Note that changing the local holomorphic coordinate z to another local holomorphic coordinate z' centered at t will change $\hat{s}(t)$ to $\hat{s}'(t) = \lim_{e\rightarrow p}(z(e)/z'(e))^{\nu_t}\hat{s}(t)$. Thus, the vector $\hat{s}(t)$ is defined up to multiplication by a nonzero complex number. In particular, $(L_s)_t$ is well-defined. By using the local triviality of W or by the argument given in [13], one can convince oneself that L_s is a holomorphic subbundle of W. Note that $L_{\lambda s} = L_s$, $\forall \lambda \in C^*$, so that we have a well-defined map $\mathbb{P}H^0(W) \longrightarrow \mathrm{Gr}^1(W)$ from the projectivisation of $H^0(W)$ to the set of holomorphic line subbundles of W.

Since s is a holomorphic section of L_s, it follows that L_s is holomorphically equivalent to $O(s)$, where $O(s)$ is the line bundle on E associated to the divisor $(s) = \sum_{t\in \mathrm{supp}(s)} \nu_t t$. In particular, we have $\mathrm{deg}L_s = \mathrm{deg}(s) = \sum_{t\in\mathrm{supp}(s)} \nu_t = \sum_{e\in E} \mathrm{deg}s(e)$, where we define $\mathrm{deg}s(e)$ to be ν_e, if $e \in \mathrm{supp}(s)$ and 0 otherwise.

For each $e \in E$, we have a natural linear map $\phi_e : H^0(W) \longrightarrow W_e$ given by $\phi_e(s) := s(e), \forall s \in H^0(W)$ (the evaluation map at e). We denote its image and kernel by $R_e := \phi_e(H^0(W)) \subset W_e$, $K_e := \mathrm{ker}\phi_e \subset H^0(W)$ and we define $r_e(W) := \dim_{\mathbb{C}} R_e$, $d_e(W) := \dim_{\mathbb{C}} K_e$. We have $r_e(W) + d_e(W) = h^0(W)$ at any point $e \in E$.

Define a subspace N_e of W_e by $N_e :=< \{\hat{s}(e)|s \in K_e\} >\subset W_e$ (if $s = 0$, we define $\hat{s}(e)$ to be zero). It is easy to see that changing the linear coordinate z does not affect N_e. Note that $N_e = \sum_{s\in K_e} (L_s)_e$. In general, the subspaces N_e, R_e of W_e may intersect and their sum need not generate W_e. Define $\mathcal{Z}(W) := \{t \in E|K_t \neq 0\}$.

If W is semistable then we must have $\mathrm{deg}s = \mathrm{deg}L_s \leq \mu(W)$. Since s is regular, we also have $\mathrm{deg}s \geq 0$. Then $\mathrm{deg}s \in \{0, .., [\mu(W)]\}$, where [ ] denotes the integer part. In particular, we have $\mathrm{deg}s(e) \leq \mu(W)$ for all $e \in E$.

Proposition 2.1 *Suppose that W is semistable and of slope 1 . Let $e \in E$ and fix a local coordinate z around e on E. Then the map $s \in K_e \rightarrow \hat{s} \in N_e$ is a $\mathbb{C}$-linear isomorphism. In particular, we have* $\dim_{\mathbb{C}} N_e = d_e$.

Proof: By the above, we see that any $s \in K_e - \{0\}$ must have a *simple* zero at e. If $s_1, s_2 \in K_e$ and $\alpha_1, \alpha_2 \in \mathbb{C}$, let $s := \alpha_1 s_1 + \alpha_2 s_2$. Then $\exists \lim_{e' \to e} z^{-1} s(e') = \alpha_1 \hat{s}_1(e) + \alpha_2 \hat{s}_2(e)$. If $\alpha_1 \hat{s}_1(e) + \alpha_2 \hat{s}_2(e) = 0$, then s must be zero (otherwise s would have degree > 1 at e). In this case $\hat{s}(e) = 0$ by definition. If $\alpha_1 \hat{s}_1(e) + \alpha_2 \hat{s}_2(e) \neq 0$, then $\hat{s}(e) = \alpha_1 \hat{s}_1(e) + \alpha_2 \hat{s}_2(e)$.

Thus in both cases we have $\hat{s}(e) = \alpha_1 \hat{s}_1(e) + \alpha_2 \hat{s}_2(e)$, which shows linearity. If $s \in K_e$, then by definition $\hat{s}(e)$ is zero only if $s = 0$. This shows injectivity. Surjectivity is obvious. $\square$

What follows is a generalization of the previous classical discussion.

Definition 2.1 *Let $s \in H^0(W)$ be a holomorphic section of W. Consider the holomorphic section $\overline{s}$ of the quotient bundle W/T, naturally induced by s. We say that s is* incident of order (degree) d on T at a point $e \in E$ *if $\overline{s}$ has a zero of order (exactly) d at e. In this case, we write $\deg_T s(e) := d$ and we call it* the incidence order(degree) *of s on T at e.*

Note that we have $s(e) \in T_e$ iff $\deg_T s(e) > 0$. Intuitively, $\deg_T s(e)$ characterizes 'how fast' $s(e') \in W_{e'}$ approaches the subspace $T_{e'}$ of $W_{e'}$ as e' approaches e on E.

If $s \in H^0(T) \subset H^0(W)$, then $\overline{s}$ is identically zero, so the degree of incidence of s on T is not defined for such s at any point of E. If $s \in H^0(W) - H^0(T)$, then $\overline{s}$ is a nonzero section of W/T and the associated divisor $(\overline{s})$ is a finite set of points of E. Therefore, the set $Z_T(s) := \{e \in E | \deg_T s(e) > 0\} = \{e \in E | s(e) \in T\} = \mathrm{supp}(\overline{s})$ is finite for all sections $s \in H^0(W) - H^0(T)$. In particular, $\deg_T s(e)$ is well defined in this case at all points $e \in E$. Thus, for all $s \in H^0(W) - H^0(T)$, we can define the *total degree of s along T* by $\deg_T s := \sum_{e \in Z_s(T)} \deg_T s(e) = \deg \overline{s}$.

For $T = \mathbf{0}$ (the null subbundle of W) we have $\overline{s} = s$ so $\deg_{\mathbf{0}} s(e) = \deg s(e)$ and the above definition reduces to the usual one.

Proposition 2.2 *Let M be a holomorphic subbundle of T and $s \in H^0(W) - H^0(T)$. Let $q \in E$ an arbitrary point. Let σ be the section of W/M induced by s via the canonical projection $W \xrightarrow{p} W/M$. Then $\deg_T s(q) = \deg_{T/M} \sigma(q)$*

Proof: Oviously T/M is a subbundle of W/M and $\sigma \in H^0(W/M) - H^0(T/M)$. s and σ induce the same section $\overline{s}$ of W/T via the canonical projections $W \to W/T$ and $W/M \to (W/M)/(T/M) \approx W/T$. Therefore: $\deg_{T/M} \sigma(q) = \deg \overline{s}(q) = \deg_T s(q)$. $\square$

We have the following :

Proposition 2.3 *Suppose that W is semistable of normalized degree $\mu(W)$ and that T has normalized degree $\mu(T) = \mu(W)$. Then we have $\deg_T s \leq \mu(W)$ for all $s \in H^0(W) - H^0(T)$.*

Proof: Indeed, T is in this case obviously semistable (since W is semistable) and thus W/T is semistable of normalized degree $\mu(W/T) = \mu(W)$ (see, for example Proposition

8 on page 18 of [15]). Then $\overline{s}$ must have have total degree at most equal to $\mu(W/T) = \mu(W)$ in order for $L_{\overline{s}} \approx O(s)$ not to destabilize W/T. Then use $\deg_T s = \deg\overline{s}$. $\Box$

For W semistable and $\mu(T) = \mu(W) = 1$, this shows that a section $s \in H^0(W) - H^0(T)$ either does not intersect T or intersects it at exactly one point, the incidence degree of s at that point being exactly one.

We now give an alternative description of the incidence degree, which is more practical from a computational point of view.

Proposition 2.4 *Let $s \in H^0(W) - H^0(T)$ and $q \in E$. Consider a local holomorphic frame $(s_1 ... s_{r_0})$ of T around q. Then*

$$\deg s(q) = \deg s \wedge s_1 \wedge ... \wedge s_{r_0}(q) \tag{6}$$

Proof: Let U be an open neighborhood of q such that the exact sequence

$$0 \longrightarrow T|_U \xrightarrow{j} W|_U \xrightarrow{p} (W/T)|_U \longrightarrow 0 \tag{7}$$

splits in the holomorphic category. Let $u : (W/T)|_u \longrightarrow W|_U$ be a holomorphic injection such that $W|_U = j(T|_U) \oplus u((W/T)|_U)$. We identify T with $j(T)$ via j and $(W/T)|_U$ with $u((W/T)|_U)$ via u. We can assume that U is small enough so that all 3 bundles involved are trivial above U. Let $s_1 ... s_{r_0}$ be a local holomorphic frame of T above U and $s_{r_0+1} ... s_r$ a frame of $(W/T)|_U \equiv u((W/T)|_U)$. Then $s_1 ... s_r$ is a local frame of W above U.

Write $s(e) = \sum_{i=1..r} f_i(e) s_i(e)$ with $f_i \in \mathcal{O}_U$. Then

$$\overline{s}(e) = \sum_{i=r_0+1...r} f_i(e) s_i(e). \tag{8}$$

and

$$s(e) \wedge s_1(e) \wedge ... \wedge s_{r_0}(e) = \sum_{i=r_0+1...r} f_i(e) s_i(e) \wedge s_1(e) \wedge ... \wedge s_{r_0}(e) \tag{9}$$

The statement $\deg_T s(q) = d$ is equivalent to $\exists \lim_{e \to q} z^{-d} \overline{s}(e) \neq 0$, which is equivalent to $\exists \lim_{e \to q} z^{-d} \overline{f}(e) \neq 0$, where $\overline{f} := (f_{r_0+1} ... f_r) \in \oplus_{i=r_0+1...r} \mathcal{O}_U$. This in turn is equivalent to $\exists \lim_{e \to q} z^{-d} s(e) \wedge s_1(e) \wedge ... \wedge s_{r_0}(e) \neq 0$. $\Box$

Now let $s \in H^0(W) - H^0(T)$ and $q \in E$. The associated section $\overline{s} \in H^0(W/T)$ defines a line subbundle $L_{\overline{s}} \subset W/T$ as above. In particular, at the point q we have a 1-dimensional subspace $(L_{\overline{s}})_q$ of the fibre $(W/T)_q = W_q/T_q$. We define $W_s(q)$ to be the $(r_0 + 1)$-dimensional subspace of W_q which induces $(L_{\overline{s}})_q$, i.e. the preimage of $(L_{\overline{s}})_q$ via the natural surjection $W_q \xrightarrow{p_q} W_q/T_q$. For $e \in E - Z_T(s)$ we obviously have $W_s(q) =< s(e) > \oplus T_e$. The following gives an analogue of this decomposition for points $q \in Z_T(s)$:

Proposition 2.5 *Let $s \in H^0(W) - H^0(T)$ and $q \in E$. Let z be any local holomorphic coordinate on E, centered at q.*

The following are equivalent :

(a) $\deg_T s(q) = d$

(b) There exist local *holomorphic sections* $\tilde{s}$ *of* W *around* q *and* s_0 *of* T *around* q *such that :*

(b1) $s(e) = z^d \tilde{s}(e) + s_0(e)$ *for all* e *sufficiently close to* q

(b2) $\tilde{s}(q) \in W_q - T_q$

In this case, we have $W_s(q) =< \tilde{s}(q) > \oplus T_q$.

Proof: Assume (a) holds and consider a neighborhood U of q such that the sequence 7 splits. Since $\deg \overline{s}(q) = d$, we can choose U small enough so that there exists a holomorphic section σ of W/T above U such that $\overline{s}(e) = z^d \sigma(e), \forall e \in U$ and $\sigma(q) \neq 0$. Then there exists a holomorphic section $\tilde{s} := u \circ \sigma$ of $W|_U$, such that $\overline{\tilde{s}} = p(\tilde{s}) = \sigma$. Thus $p(s - z^d \tilde{s}) = 0$, so that $s(e) - z^d \tilde{s}(e) \in T_e$, $\forall e \in U$. Since $s(e) - z^d \tilde{s}(e)$ is holomorphic, this gives a holomorphic section s_0 of $T|_U$ such that $s = z^d \tilde{s} + s_0$ and $(b1)$ holds. Moreover, $\sigma(q) \neq 0$ implies $\tilde{s}(q) \in W_q - T_q$ and thus $(b2)$ holds. Since $\overline{s}(e) = z^d \sigma(e)$, we have $\hat{\overline{s}}(q) = \sigma(q)$ so that $\sigma(q) \in (L_{\overline{s}})_q$. Thus $\tilde{s}(q) \in p_q^{-1}((L_{\overline{s}})_q) = W_s(q)$ and $W_s(q) =< \tilde{s}(q) > \oplus T_q$.

The converse implication is trivial in view of the previous proposition. □

Note that $\tilde{s}$, s_0 cannot, in general, be extended beyond a neighborhood of q. Also note that $\tilde{s}(q)$ is only determined modulo T_q and modulo a constant multiplicative factor (from the choice of the local holomorphic coordinate z around q).

Definition 2.2 *Let* $s \in H^0(W) - H^0(T)$. *Define* $W_{s,T} = \sqcup_{e \in E} W_s(e)$. *Then* $W_{s,T}$ *has a natural structure of holomorphic vector bundle over* E *and* $s \in H^0(W_{s,T})$ *while* T *is a holomorphic subbundle of* $W_{s,T}$.

Proof: A holomorphic trivialization of $W_{s,T}$ is obtained as follows. For U an open set such that $U \cap Z_T(s) = \Phi$, choose a local frame $s_1..s_{r_0}$ of T over U and trivialize $W_{s,T}$ over U by using the local frame $s_1...s_{r_0}, s$. For U such that $U \cap Z_T(s) = q$ (a single point), by choosing U small enough and picking a local holomorphic coordinate z on E, one one can write $s(e) = z^d \tilde{s}(e) + s_0(e)$ as before, where $d = \deg_T(s)$ and $\tilde{s}$ does not meet T over U. Then $\tilde{s}$ is a local holomorphic section of W above U and one can trivialize $W_{s,T}$ over U by using $s_1...s_{r_0}, \tilde{s}$, where $s_1...s_{r_0}$ is a local holomorphic frame of T above U. The holomorphic compatibility of the various local trivializations is immediate. □

Intuitively, the fibre $W_{s,T}(q) =< \tilde{s}(q) > \oplus T_q$ for $q \in Z_T(s)$ is the correct 'limit' of the fibres $W_{s,T}(e) =< s(e) > \oplus T_e$ as $e \to q$. The section s determines a line subbundle L_s of W and we have $W_{s,T} = L_s \oplus T$. For $T = \mathbf{0}$ (the null subbundle of W), we obviously have $W_{s,\mathbf{0}} = L_s$. This is a generalization of the construction of L_s.

Now suppose that the set $\mathcal{Z}(W)$ is finite [5]. In this case, if $s_1...s_k \in H^0(W)$ are $\mathbb{C}$-linearly independent sections of W, then they are also $\mathbb{C}$-linearly independent at the generic point of E (i.e. $s_1(e)...s_k(e)$ are linearly independent in W_e for a generic

[5] We will see that this is the case if $W = V \otimes O(p)$ with V semistable and of degree zero

$e \in E$); then we can define inductively $W_{s_1...s_k} := W_{s_k, W_{s_1...s_{k-1}}}$, with $W_{s_1} := L_{s_1}$. Indeed, one can easily see that $s_2 \in H^0(W) - H^0(W_{s_1})$ and (by induction) $s_j \in H^0(W) - H^0(W_{s_1...s_{j-1}})$, $\forall j = 2..k$, due to the generic linear independence of $s_1..s_k$. $W_{s_1...s_k}$ is a rank k vector bundle and $s_1..s_k$ are sections of $W_{s_1..s_k}$ which are linearly independent at the generic point. Intuitivley, $W_{s_1...s_k}$ is the subbundle of W 'spanned' by $s_1..s_k$.

If $(\sigma_1...\sigma_k)^t = A(s_1...s_k)^t$, with $A \in GL(k, \mathbb{C})$ a constant nondegenerate matrix, then it is easy to see that $W_{\sigma_1..\sigma_k} = W_{s_1...s_k}$. Indeed, $s_1...s_k$ are sections of $W_{\sigma_1..\sigma_k}$ (since $\sigma_1...\sigma_k$ are) so that $W_{\sigma_1...\sigma_k,e} = W_{s_1..s_k,e}$ for all e with $s_1(e)...s_k(e)$ linearly independent. For $q \in E$ such that $s_1(q)...s_k(q)$ are linearly dependent, one can consider vectors $\tilde{\sigma}_1(q), .., \tilde{\sigma}_k(q)$, determined by local sections σ_j of $W_{\sigma_1...\sigma_j}$ and σ_{0j} of $W_{\sigma_1...\sigma_{j-1}}$ via the conditions: $\sigma_j(e) = z^{deg_{W_{\sigma_1...\sigma_{j-1}}}\sigma_j(q)}\tilde{\sigma}_j(e) + \sigma_{0j}(e)$ for e close to q and $\sigma_j(q) \in W_{\sigma_1..\sigma_j,q} - W_{\sigma 1...\sigma_{j-1},q}$. Note that $\tilde{\sigma}_1(q), .., \tilde{\sigma}_k(q)$ are linearly independent. These vectors obviously belong to $W_{s_1...s_k}(q)$ since for $e \neq q$ they are related to $\sigma_1...\sigma_k$ (and thus to $s_1...s_k$) by linear combinations of these vectors and since subbundles of W are closed in the total space of W.

Thus $W_{\sigma_1...\sigma_k}(q) =< \tilde{\sigma}_1(q), .., \tilde{\sigma}_k(q) >\subset W_{s_1..s_k}(q)$ and they must coincide since they have the same dimension. Therefore, $W_{s_1...s_k}$ depends only on the subspace $< s_1...s_k >$ of $H^0(W)$. Thus, if $\mathcal{Z}(W)$ is finite then we have a natural map :

$$\psi_k : \mathrm{Gr}^k(H^0(W)) \to \mathrm{Gr}^k(W). \tag{10}$$

An alternative way to understand this is as follows (cf. [13]). If $\mathcal{Z}(W)$ is finite, then given a k-dimensional subspace K of $H^0(W)$, $\phi_e(K)$ defines a rational section f of $\mathbb{G}\mathrm{r}^k(W)$, where $\mathbb{G}\mathrm{r}^k(W)$ is the bundle obtained by taking the grassmannian $\mathrm{Gr}^k(W_e)$ of W_e as the fibre above each $e \in E$. Singularities of this section may appear only at a point e where $\phi_e(K)$ fails to be k-dimensonal, i.e. at the points $e_1..e_s$ of E where the values of a system $s_1..s_k$ of sections of W giving a basis of K fail to be linearly independent. Loosely speaking, one may worry that at such points there is no 'completion' of the set $\{\phi_e(K) | e \in E - \{e_1..e_s\}\}$ which makes it into the total space of a holomorphic vector bundle. This does not happen for the following reason. With the natural structure, $\mathbb{G}\mathrm{r}^k(W)$ is a complete variety and a classical result implies that f must be regular. Thus f determines a subbundle of W, which clearly coincides with $W_{s_1...s_k}$.

Again assuming $\mathcal{Z}(W)$ to be finite, suppose that we are given a filtration $\mathcal{K} : 0 := K_0 \subset K_1 \subset ... \subset K_{k-1} \subset K_k$ of a subspace K_k of $H^0(W)$, such that $\dim_{\mathbb{C}} K_j = j$, $\forall j = 0..r$. Associated to $\mathcal{K}$ via ψ there is a filtration $\mathcal{W}(\mathcal{K}) : 0 := W_0 \subset W_1 \subset ... \subset W_{k-1} \subset W_k$ by holomorphic subbundles with $\mathrm{rank} W_j = j$, $\forall j = 0..r$. If $s_j \in K_j - K_{j-1}$ for all $j = 1..k$, then it is obvious that the integers $\delta_j^{\mathcal{K}}(t) := \deg s_1 \wedge ... \wedge s_j(t)$ ($t \in \mathcal{Z}(W), j = 1..r$) depend only on $\mathcal{K}$. It is also easy to see – by using Proposition 2.5 – that $\deg_{W_{j-1}} s_j(t) = \delta_j^{\mathcal{K}}(t) - \delta_{j-1}^{\mathcal{K}}(t)$, where we let $\delta_0^{\mathcal{K}}(t)$ be equal to 0.

2.2 The space of sections of the bundles F'_r

Let $F'_r := F_r \otimes O(p)$. F'_r is a rank r indecomposable and semistable bundle of slope 1. Since F_r is semistable and of degree zero, we have $h^0(F'_r) = r$ and $h^1(F'_r) = 0$. Recall from [13] that we have exact sequences:

$$0 \longrightarrow F_k \xrightarrow{i} F_r \xrightarrow{p} F_l \longrightarrow 0 \tag{11}$$

for all $k, l \geq 0$ with $k + l = r$. Below we will use their twisted version :

$$0 \longrightarrow F'_k \xrightarrow{i} F'_r \xrightarrow{p} F'_l \longrightarrow 0 \tag{12}$$

For $l = 1$, we obtain the twisted version of the defining sequences of F'_r:

$$0 \longrightarrow F'_{r-1} \xrightarrow{i} F'_r \xrightarrow{p} O(p) \longrightarrow 0 \tag{13}$$

while for $k = 1$ this gives :

$$0 \longrightarrow O(p) \xrightarrow{j} F'_r \xrightarrow{p} F'_{r-1} \longrightarrow 0 \tag{14}$$

Since $H^1(F'_k) = 0$, the exact cohomology sequence associated to (12) collapses to:

$$0 \longrightarrow H^0(F'_k) \xrightarrow{i_*} H^0(F'_r) \xrightarrow{p_*} H^0(F'_l) \longrightarrow 0 \tag{15}$$

Being an exact sequence of vector spaces, this must split. Therefore, there must exist $\mathbb{C}$-bases $< \sigma_1 ... \sigma_k >$ of $H^0(F'_k)$, $< \sigma_{k+1} ... \sigma_r >$ of $H^0(F'_l)$ and $< s_1 ... s_r >$ of $H^0(F'_r)$ such that $j_*(\sigma_i) = s_i, \forall i = 1..k$ and $p_*(s_j) = \sigma_j, \forall j = k+1..r$.

Proposition 2.6 *For any $r \geq 1$, we have $d_p(F'_r) = 1$ and $d_e(F'_r) = 0$ for all $e \in E - \{p\}$.*

Proof: The sequence (14) shows that $d_p(F'_r) > 0$.

Now suppose that $d_p(F'_r) > 1$. Then there exist two linearly independent sections s_1, s_2 of F'_r such that $s_1(p) = s_2(p) = 0$. Let $L_i = L_{s_i}$ be the associated line subbundles of F'_r. Since F'_r is semistable and of degree 1, we must have $\deg L_i = 1$ and $(s_i) = p_i$. Hence $\exists \lim_{e \to q} s_i(e)/z = \hat{s}_i(q) \neq 0$.

Suppose that $\hat{s}_1(p), \hat{s}_2(p)$ are linearly dependent. Then we can write $\hat{s}_1(p) = \alpha \hat{s}_2(p)$ with $\alpha \in \mathbb{C}^*$. The section $s := s_1 - \alpha s_2$ is then nonzero (since s_1, s_2 are $\mathbb{C}$-linearly independent) and we obviously have $\deg s(p) \geq 2$, which contradicts semistability of F'_r. Thus, it must be the case that $\hat{s}_1(q), \hat{s}_2(q)$ are linearly independent.

Now suppose there exists $e_0 \in E - \{p\}$ such that $s_1(e_0)$ and $s_2(e_0)$ are linearly dependent. Write $s_1(e_0) = \beta s_2(e_0)$, with $\beta \in \mathbb{C}^*$. Then the section $s' = s_1 - \beta s_2$ vanishes both at e_0 and at p and so $\deg s'(p) \geq 2$, again contradicting semistability of F'_r. It follows that $s_1(e), s_2(e)$ are linearly independent for all $e \in E - \{p\}$.

From these two facts we immediately see that the subbundle sum $L_1 + L_2$ is *direct*. Since $(s_i) = p$, we also have $L_i \approx O(p)$; thus we have a holomorphic subbundle $L_1 \oplus$

$L_2 = O(p) \oplus O(p)$ of F'_r. Twisting by $O(-p)$, this gives a trivial subbundle of rank two $I_2 \subset F_r$. Since $h^0(I_2) = 2$, this would imply $h^0(F_r) \geq 2$, a contradiction. This finishes the proof of the first statement.

Now let $e \in E - \{p\}$. To show that $K_e = 0$, we proceed by induction on r, using the sequence (14).

For $r = 1$ the statement is obvious. Suppose the statement holds for $r - 1$, but fails for r. Then there exists a nonzero section s of F'_r such that $s(e) = 0$. We cannot have $s \in H^0(\mathrm{im} j)$ since that would imply $\deg s \geq 2$ (as $e \neq p$), which contradicts semistability of F'_r. Thus $\overline{s} := p_*(s)$ is a nonzero section of F'_{r-1}. Since $s(e) = 0$, we have $\overline{s}(e) = 0$, so that $K_e(F'_{r-1}) \neq 0$. This is impossible by the induction hypothesis. $\square$

Consider the commutative group structure $(E, \oplus)$ on E with zero element p. If $q_1, q_2 \in E$, then $q_1 \oplus q_2$ is defined to be the unique point q of E such that $(q_1) + (q_2) \sim (q) + (p)$, i.e. $O(q_1 + q_2) \approx O(q + p)$. Thus $(q_1 \oplus q_2) \sim (q_1) + (q_2) - (p)$. Then $(q_1 \oplus ... \oplus q_r) \sim (q_1) + ... + (q_r) - (r - 1)p$.

Let $T_r^{(p)}(E)$ be the r-torsion subgroup of (E, p), i.e. the set of points $t \in E$ such that $rt = 0$ in $(E, \oplus)$, which is equivalent to $r(t) \sim r(p)$, i.e $O(rt) \approx O(rp)$. The map $q \in E \to O(q - p) \in \mathrm{Pic}^0(E)$ is a group isomorphism from $(E, \oplus)$ to $\mathrm{Pic}^0(E)$, which maps $T_r^{(p)}(E)$ to the subgroup $U_r := \{L \in \mathrm{Pic}^0(E) | L^r \approx O_E\} \subset \mathrm{Pic}^0(E)$ of roots of order r of O_E. We have $U_r \approx (\mathbb{Z}_r)^2$.

Proposition 2.7 *Let $r > 0$. The isomorphism classes of* indecomposable *bundles A' which can be presented as extensions :*

$$0 \longrightarrow I_{r-1} \xrightarrow{j} A' \xrightarrow{p} O(rp) \longrightarrow 0 \tag{16}$$

of $O(rp)$ by the trivial rank $r - 1$ bundle I_{r-1} are in bijective correspondence with U_r. More precisely, each such bundle A' is of the form:

$$A' = O(q) \otimes F_r = L \otimes F'_r \tag{17}$$

where $q \in T_r^{(p)}(E)$ and $L := O(q - p) \in U_r$. Here $F'_r := F \otimes O(p)$.

Note that we are *not* considering extension classes, but isomorphism classes of bundles which can be presented as extensions.

Proof:

Show that F'_r fit into such sequences

Use induction on r. For $r = 1$ the statement is obvious (with $I_0 = \mathbf{0}$). Suppose the statement holds for $r - 1$, so that there is an exact sequence:

$$0 \longrightarrow I_{r-2} \xrightarrow{j} F'_{r-1} \xrightarrow{p} O((r - 1)p) \longrightarrow 0 \tag{18}$$

Let $s_1 ... s_{r-1}$ be a basis of $H^0(I_{r-2})$ and s_{r-1} a section of F'_{r-1} such that $s_1 ... s_{r-1}$ is a basis of $H^0(F'_{r-1})$ and (using 15) such that $p_*(s_{r-1}) \in H^0(O(p)) - \{0\}$. Then

$s_1(e)...s_{r-2}(e)$ are linearly independent for all $e \in E$ By Proposition 2.6, we have that $s_{r-1}(e) \in F'_{r-1,e} - < s_1(e)...s_{r-2}(e) >= F'_{r-1,e} - I_{r-1,e}$ for all $e \neq p$. Now use the recursive definition (13) of F'_r. This shows that we can choose $s_r \in H^0(F'_r)$ such that $s_1...s_r$ is a basis of $H^0(F'_r)$ and such that the induced section $\overline{s}_r \in H^0(O(p))$ has zero divisor $(\overline{s}_r) = (p)$. Since $s_{r-1}(p) = 0$, Proposition 2.6 applied to F'_r shows that $s_1(e)...s_{r-2}(e), s_r(e)$ are linearly independent for *all* $e \in E$, while $s_1(e)...s_{r-2}(e), s_{r-1}(e)$, $s_r(e)$ are linearly independent for $e \neq p$. Thus $s_1(e)...s_{r-2}(e), s_r(e)$ determine a trivial subbundle I_{r-1} of F'_r and $s_{r-1}(e)$ belongs to this subbundle iff $e = p$ (where $s_{r-1}(p) = 0$). It follows that the induced section $\overline{s}_{r-1}$ of the line bundle $L := F'_r/I_{r-1}$ vanishes only at p. Since $\deg F'_r = r$, we have $\deg L = r$ so that $\deg(\overline{s}_{r-1}) = r$. Therefore $(\overline{s}_{r-1}) = rp$ and $L \approx O(rp)$. This gives an exact sequence :

$$0 \longrightarrow I_{r-1} \xrightarrow{j} F'_r \xrightarrow{p} O(rp) \longrightarrow 0 \tag{19}$$

Show that $O(q) \otimes F_r$ for $q \in T_r^{(p)}(E)$ are also extensions of $O(rp)$ by I_{r-1}

Since $q \in T_r^{(p)}(E)$, we have $O(rq) \approx O(rp)$. Combined with (19)(applied for p substituted with q), this gives the desired statement.

Show that any indecomposable A' which can be presented as such an extension is of this form

If A' is an extension of $O(rp)$ by I_{r-1}, then $\det A' \approx O(rp)$. If A' is indecomposable then $A := A' \otimes O(-p)$ belongs to $\mathcal{E}(r, 0)$, so that $A \approx O(q-p) \otimes F_r$ for some $q \in E$. (q is uniquely determined by A). Then $A' \approx O(q) \otimes F_r$, so that $\det A' \approx O(rq)$. Thus we must have $O(rq) \approx O(rp)$ i.e. $q \in T_r^{(p)}(E)$. This finishes the proof. □

Theorem 2.1 *Let V be a degree zero holomorphic vector bundle of rank r over E and let $V' := V \otimes O(p)$. The following statements are equivalent :*

(a) V is holomorphically equivalent to $O(q) \otimes F_r$, where q is a point of E

(b) There exists a $\mathbb{C}$-basis $(s_1...s_r)$ of $H^0(V')$ with the following properties :

(b1) $s_1(e)...s_r(e)$ is a basis of V'_e for all $e \in E - \{p\}$

(b2) $s_1(p) = 0$ and $s_2(p), ..., s_r(p)$ are linearly independent in V'_p

(b3) $\deg s_1 \wedge s_2 \wedge ... \wedge s_j(p) = j$ for all $j = 1..r$.

(c) The following conditions are satisfied :

(c1) $h^0(V') = r$

(c2) $\mathcal{Z}(V') = \{p\}$

(c3) There exists a nondegenerate *filtration*

$$\mathcal{K} : 0 = K_0 \subset K_1 \subset ... \subset K_r := H^0(V') \tag{20}$$

of $H^0(V')$, with associated filtration

$$0 = W_0 \subset W_1 \subset ... \subset W_r := V' \tag{21}$$

of V', having the properties :

(c31) $K_j = \{s \in H^0(V') | s(p) \in (W_{j-1})_p\}$ (i.e. $K_j = \phi_p^{-1}(W_{j-1})$), $\forall j = 1..r$

(c32) $\delta_j^{\mathcal{K}}(p) = j, \forall j = 1..r$

Moreover, in this case we have $W_j \approx F'_j$ *and* $K_j = H^0(W_j) \approx H^0(F'_j)$ *for all* $j = 1..r$.

Note that $s_2...s_r$ generate a trivial subbundle I_{r-1} of F'_r (since they are everywhere linearly independent), while the section s_1 is incident on I_{r-1} at p in order r. This is in agreement with the previous proposition. The precise manner of incidence of s_1 on I_{r-1} is controlled by condition $(b3)$.

Note that $(c31)$ acts as an inductive definition of the filtration $\mathcal{K}$. For $j = 1$, $(c31)$ gives $K_1 = \ker\phi_p = K_p(V')$. The map $\psi_1 : \mathrm{Gr}^1(H^0(V')) \to \mathrm{Gr}^1(V')$ gives the subbundle $W_1 = \psi_1(K_1)$. Then $(c32)$ for $j = 2$ defines K_2, the map ψ_2 gives $W_2 = \psi_2(K_2)$ and so on. In particular, $\mathcal{K}$ is naturally associated to F'_r [6]. It is easy to see from the proof of the theorem below that $\mathcal{K}$ is nothing other then the cohomology filtration induced by the standard Jordan-Holder filtration of F'_r :

$$0 \longrightarrow F'_1 \longrightarrow F'_2 \longrightarrow ... \longrightarrow F'_{r-1} \longrightarrow F'_r \tag{22}$$

Indeed, (22) has the partial sequences:

$$0 \longrightarrow F'_{j-1} \longrightarrow F'_j \longrightarrow O(p) \longrightarrow 0 \tag{23}$$

(for $j = 2...r$). Since $H^1(F'_{j-1}) = 0, \forall j = 2..r$, these give the exact sequences :

$$0 \longrightarrow H^0(F'_{j-1}) \longrightarrow H^0(F'_j) \longrightarrow H^0(O(p)) \longrightarrow 0 \tag{24}$$

which combine to give the filtration :

$$0 \longrightarrow H^0(F'_1) \longrightarrow H^0(F'_2) \longrightarrow ... \longrightarrow H^0(F'_{r-1}) \longrightarrow H^0(F'_r) \tag{25}$$

of $H^0(F'_r)$. This can be identified with the filtration $\mathcal{K}$ in the theorem.

Proof:

Show that (a) implies (b)

We proceed by induction on r. For $r = 1$, the statement is trivial. Let $r \geq 2$ and suppose the statement holds for $r - 1$. By the above discussion, we can choose bases $\sigma_1...\sigma_{r-1}$ of $H^0(F'_{r-1})$, σ_r of $H^0(O(p))$ and $s_1...s_r$ of $H^0(F'_r)$ such that $j_*(\sigma_1) = s_1...j_*(\sigma_{r-1}) = s_{r-1}$ and $p_*(s_r) = \sigma_r$.

Since the result holds for $r - 1$, we can further assume that $\sigma_1...\sigma_{r-1}$ satisfy the properties (b) for r replaced with $r - 1$. Since $p_*(s_r) = \sigma_r$ and $(\sigma_r) = p$, it folows that $s_r(e) \in (F'_r)_e - j_e((F'_{r-1})_e), \forall e \in E - \{p\}$, while $s_r(p) \in j_p((F'_{r-1})_p)$. Since j_e is injective for all $e \in E$, and since $\sigma_1...\sigma_{r-1}$ satisfy $(b1)$, we see that $s_1(e)...s_r(e)$ are linearly independent for all $e \in E - \{p\}$, so that $s_1...s_r$ satisfy $(b1)$. By $(b2)$ for $\sigma_1...\sigma_{r-1}$ we obtain that $s_1(p) = 0$ and $s_2(p)...s_{r-1}(p)$ are linearly independent.

[6] Of course, F'_r are only determined up to isomorphism. Naturality heer means that such an isomorphism is compatible with the filtrations $\mathcal{K}$

Now suppose that $s_r(p) \in < s_2(p)...s_{r-1}(p) >$. Then $s_r(p) = \alpha_2 s_2(p) + .. + \alpha_{r-1}s_{r-1}(p)$. Then $s := s_r - \alpha_2 s_2 - .. - \alpha_{r-1}s_{r-1}$ is a regular section of F'_r which vanishes at p. Since s_r is linearly independent of $s_1...s_{r-1}$, it is clear that s is linearly independent of $s_1...s_{r-1}$. In particular, s is linearly independent of s_1. This implies that we have two linearly independent sections s_1, s of F'_r, both vanishing at p. Since this is impossible by virtue of Proposition 2.6, it follows that $s_2(p)...s_r(p)$ are linearly independent and $(b2)$ holds.

Since $p_*(s_r) = \sigma_r$ has a simple zero at p, it follows that s_r vanishes in order 1 along the subbundle $j_*(F'_{r-1})$ of F'_r. Since $(b3)$ holds for F'_{r-1} by the induction hypothesis, we also know that s_j vanishes in order 1 along the subbundle W_j of F'_{r-1}, where $W_j = W_{s_1...s_j}$, for all $j = 1..r-1$. In particular, we have $s_j(e) = z\tilde{s}_j(p) + s_{0j}(e)$, with $s_{0j} \in H^0(W_{j-1})$ for all $j = 1..r-1$, and all e sufficiently close to p. This implies that $s_1(e) \wedge ... \wedge s_{r-1}(e) = z^{r-1}\tilde{s}_1(e) \wedge ... \wedge \tilde{s}_{r-1}(e)$ so that $\tilde{s}_1(e) \wedge ... \wedge \tilde{s}_{r-1}(e) \neq 0$ for e near p. This shows that $\tilde{s}_1, ..., \tilde{s}_{r-1}$ give a local holomorphic frame of F'_{r-1} in a vicinity of p. Then by Proposition 6, we must have $\deg \tilde{s}_1 \wedge ... \wedge \tilde{s}_{r-1} \wedge s_r(p) = 1$, so that $\deg s_1 \wedge ... \wedge s_r(p) = r$. Thus $(b3)$ holds for F'_r. Thus (a) implies (b).

Show that (b) implies (c)

Assume (b) holds. Then $(c1)$ and $(c2)$ are obvious. We can construct a filtration:

$$\mathcal{K} : 0 := K_0 \subset K_1 := < s_1 > \subset K_2 := < s_1, s_2 > \subset ... \subset K_r := H^0(V') \tag{26}$$

of $H^0(V')$, and an associated filtration :

$$\mathcal{W} : 0 := W_0 \subset W_1 \subset W_2 \subset ... \subset W_r := V' \tag{27}$$

of V', as explained in the previous subsection. Let us analyze the situation at the point p.

Claim: For each $j = 1..r$, we have $\deg_{W_{j-1}} s_j(p) = 1$ and $s_2(p)...s_j(p)$ is a $\mathbb{C}$-basis of $(W_{j-1})_p$.

We prove the claim by induction on j. For $j = 1$ we have $W_{j-1} = W_0 = \mathbf{0}$ and, by $(b3)$, we have $\deg_{W_0} s_1(p) = \deg s_1(p) = 1$. The second part of the claim is trivial in this case.

Now let $j \in \{2...r\}$ and assume that the claim is true for all $j' < j$. Fix a local coordinate z on E, centered at p. By Proposition 2.5, we can write :

$$s_k(e) = z\tilde{s}_k(e) + s_{0k}(e),\ \forall k = 1..j-1 \tag{28}$$

for all e sufficiently close to p, where $\tilde{s}_k(p) \in V'_p - (W_{k-1})_p$ and s_{0k} is a local section of W_{k-1}. Then $s_1(e) \wedge ... \wedge s_{j-1}(e) = z^{j-1}\tilde{s}_1(e) \wedge ... \wedge \tilde{s}_{j-1}(e)$ for e close to p. By $(b3)$, we have $\tilde{s}_1(p) \wedge ... \wedge \tilde{s}_{j-1}(p) \neq 0$ and by continuity $\tilde{s}_1(e) \wedge ... \wedge \tilde{s}_{j-1}(e) \neq 0$ for e close to p. Thus $\tilde{s}_1...\tilde{s}_{j-1}$ is a local holomorphic frame of W_{j-1} around p. We obtain :

$$s_1(e) \wedge ... \wedge s_j(e) = z^{j-1}\tilde{s}_1(e) \wedge ... \wedge \tilde{s}_{j-1}(e) \wedge s_j(e) \tag{29}$$

(for e close to p), which together with $(b3)$ gives :

$$\deg \tilde{s}_1 \wedge ... \wedge \tilde{s}_{j-1} \wedge s_j(p) = 1 \tag{30}$$

Since $\tilde{s}_1 ... \tilde{s}_{j-1}$ is a local holomorphic frame of W_{j-1} around p, this shows, by Proposition 6, that $\deg_{W_{j-1}} s_j(p) = 1$.

Since $\tilde{s}_1(p) \wedge ... \wedge \tilde{s}_{j-1}(p) \wedge s_j(p) = 0$ by (30), it follows that $s_j(p) \in <\tilde{s}_1(p)...\tilde{s}_{j-1}(p)> = (W_{j-1})_p$. By the induction hypothesis, $s_2(p)...s_{j-1}(p)$ is a basis of $(W_{j-2})_p \subset (W_{j-1})_p$, so that $s_2(p)....s_{j-1}(p) \in (W_{j-1})_p$. Thus, the vectors $s_2(p)....s_j(p)$ all belong to the j-dimensional vector space $(W_{j-1})_p$. Since they are linearly independent by $(b2)$, they must form a basis of this subspace. This finishes the proof of the claim.

Since $\delta_j^{\mathcal{K}}(p) - \delta_{j-1}^{\mathcal{K}}(p) = \deg_{W_{j-1}} s_j(p)$, the first part of the claim implies $(c32)$. The second part of the claim is easily seen to imply $(c31)$. Thus (b) implies (c).

Show that (c) implies (a)

Again proceed by induction on r. For $r = 1$ the statement is immediate.

Now let $r > 1$ and suppose that $(c) \Rightarrow (a)$ holds for $r - 1$. Also assume that V' satisfies (c). Since $\mathcal{K}$ is nondegenerate, we have $\dim_{\mathbb{C}} K_j = j$ for all $j = 1..r$. In particular, K_1 is a line bundle. By $(c31)$ and $(c31)$ we have $K_1 \approx O(p)$. Define $W' := V'/K_1$. We have an exact sequence:

$$0 \longrightarrow W_1 \xrightarrow{j} V' \xrightarrow{p} W' \longrightarrow 0 \tag{31}$$

To show (a) it suffices to show that $W' \approx F'_{r-1}$ and that (31) is nonsplit. By the induction hypothesis, to show $W' \approx F'_{r-1}$ it suffices to show that W' satisfies (c) for $r - 1$. We proceed to do this.

Show that W' satisfies $(c1)$. Since $H^1(O(p)) = 0$, (31) gives :

$$0 \longrightarrow H^0(O(p)) \xrightarrow{j_*} H^0(V') \xrightarrow{p_*} H^0(W') \longrightarrow 0 \tag{32}$$

Thus $h^0(W') = r - 1$.

Show that W' satisfies $(c2)$. For each $e \in E - \{p\}$ we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(K_1) & \xrightarrow{j_*} & H^0(V') & \xrightarrow{p_*} & H^0(W') & \longrightarrow 0 & \\ & & \phi_e^{O(p)} \downarrow & & \phi_e \downarrow & & \phi'_e \downarrow & & \\ 0 & \longrightarrow & K_{1,e} & \xrightarrow{j_e} & V'_e & \xrightarrow{p_e} & W'_e & \longrightarrow & 0 \end{array} \tag{33}$$

where the vertical arrows represent the evaluation maps. $\phi_e^{O(p)}$ is trivially an isomorphism, while ϕ_e is an isomorphism since V' satisfies $(c1)$ and $(c2)$. Thus ϕ'_e is an isomorphism. We will see below that ϕ_p is not injective. Thus W' satisfies $(c2)$.

Show that W' satisfies $(c31)$ *and* $(c32)$. First we show that $K_j = H^0(W_j)$ for all $j = 1..r$. To see this, note that $(c31)$ implies $H^0(W_{j-1}) \subset K_j$ for all j. This inclusion is *strict* (otherwise $\phi_e|_{K_j}$ for $e \neq p$ would coincide with the evaluation map of W_{j-1}; since ϕ_e is injective and $\dim_{\mathbb{C}} K_j = j$, this would contradict the rank theorem). We also trivially have $K_j \subset H^0(W_j)$ for all j. This gives $H^0(W_{j-1}) \subset K_j \overset{\neq}{\subset} H^0(W_j)$ for all j and since $\dim_{\mathbb{C}} K_j = j$ we obtain $K_j = H^0(W_j)$.

$\mathcal{K}$ induces a filtration $\mathcal{K}'$:

$$0 = K'_0 \subset K'_1 \subset ... \subset K'_{r-1} \tag{34}$$

by $K'_j := p_*(K_{j+1})$ for all $j = 1..r-1$. By (32) we have $K'_{r-1} = H^0(W')$ and $\dim_C K'_j = j$ for all $j = 1..r-1$. On the other hand, the filtration $\mathcal{W}$ of V' induces a nondegenerate filtration $\mathcal{W}'$ of W':

$$0 = W'_0 \subset W'_1 \subset ... \subset W'_{r-1} = W' \tag{35}$$

by $W'_j := p(W'_{j+1})$.

For each $j = 1..r-1$ we have a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_1 & \xrightarrow{j_*} & K_{j+1} & \xrightarrow{p_*} & K'_j & \longrightarrow 0 & \\ & & \phi_p \downarrow & & \phi_p \downarrow & & \phi'_p \downarrow & & \\ 0 & \longrightarrow & K_{1,p} & \xrightarrow{j_p} & W_{j,p} & \xrightarrow{p_p} & W'_{j-1,p} & \longrightarrow & 0 \end{array} \tag{36}$$

(we have $\phi'_p(K'_j) = \phi'_p(p_*(K_{j+1})) = p_p(\phi_p(K_{j+1})) \subset p_p(W_{j,p}) = W'_{j-1,p}$ where we used $(c31)$ for V'). Commutativity of the second square gives $p_*^{-1}(\phi_p^{'-1}(W'_{j-1,p})) = \phi_p^{-1}(p_p^{-1}(W'_{j-1,p})) = \phi_p^{-1}(W_{j,p}) = K_{j+1}$ where we used $(c31)$ for V'. Thus $\phi_p^{-1}(W'_{j-1,p}) = p_*(K_{j+1}) = K'_j$ and $\mathcal{K}', \mathcal{W}'$ satisfy $(c31)$.

Now pick $s_j \in K_j - K_{j-1}$ for all $j = 1..r$ and let $\sigma_j := p_*(s_{j+1})$ for all $j = 1..r-1$. Then $\sigma_j \in K'_j - K'_{j-1}$ and $\deg_{W'_{j-1}}\sigma_j(p) = \deg_{W_j} s_{j+1}(p)$ by Proposition 2.2. Using $(c32)$ for V' and $\delta_j^{\mathcal{K}}(p) - \delta_{j-1}^{\mathcal{K}}(p) = \deg_{W_{j-1}}\sigma_j(p)$, this immediately implies $(c32)$ for W'. (In particular, we have $\ker\phi'_p = K'_1 \neq 0$, as announced above).

Now suppose that (31) is split. Then $V' \approx O(p) \oplus F'_{r-1}$. Since $O(p)$ and F'_{r-1} both posess nonzero sections which vanish at p, this immediately gives two linearly independent sections of V' which vanish at p. But $(c3)$ implies $d_p(V') = 1$, which gives a contradiction. Thus, (31) cannot split and we must have $V' \approx F'_r$ and $V \approx F_r$. Thus (c) implies (a).

To prove the last statement of the theorem it suffices to note that each of the bundles W_j in (c) also satisfies (c) for the appropriate rank. □

It is now possible to analyze the freedom in the choice of $\tilde{s}_j$ and define a notion of canonical bases of $H^0(F'_r)$ by imposing further conditions on $s_1..s_r$. This leads to a concrete description of the endomorphisms of F'_r via their induced action on $H^0(F'_r)$, which can then be used to analyze the endomorphisms of a general degree zero semistable bundle by using the results of the next subsection. Since this is not directly related to the main focus of the present paper, we will not proceed down that path.

2.3 The main theorem

The results of the previous subsection immediately lead to:

Theorem 2.2 *Let V be a degree zero holomorphic vector bundle of rank r over E and $V' = V \otimes O(p)$. Let ϕ_e be the evaluation map of V' and $S := \mathcal{Z}(V') := \{t \in E | K_t(V') \neq 0\}$. The following are equivalent :*

(a) V is semistable

(b0) $h^0(V') = r$

(b1) The set S is finite. Let $d_t := \dim_{\mathbb{C}} K_t(V')$ for all $t \in S$.

(b3) There exists a direct sum decomposition :

$$H^0(V') = \oplus_{t \in S} \oplus_{i=1..d_t} K^{(i)}_{r_{t,i}}(t) \tag{37}$$

with $\dim_{\mathbb{C}} K^{(i)}_{r_{t,i}}(t) = r_{t,i}$ *and* nondegenerate *filtrations :*

$$\mathcal{K}^{(i)}(t) \ : \ 0 = K^{(i)}_0(t) \subset K^{(i)}_1(t) \subset ... \subset K^{(i)}_{r_{t,i}}(t) \tag{38}$$

with ψ-associated bundle filtrations:

$$\mathcal{W}^{(i)}(t) \ : \ 0 = W^{(i)}_0(t) \subset W^{(i)}_1(t) \subset ... \subset W^{(i)}_{r_{t,i}}(t) \tag{39}$$

with the properties :

(b31) We have $V'_t = R_t(V') \oplus \oplus_{i=1..d_t} (W^{(i)}_{r_{t,i}}(t))_t$, for all $t \in S$.

(b32) $\delta^{\mathcal{K}^{(i)}(t)}_s(t) = s$ for all $t \in S$, all $i = 1..d_t$ and all $s = 1..r_{t,i}$

(b33) The induced filtrations $0 = \phi_t(K^{(i)}_1(t)) \subset ... \subset \phi_t(K^{(i)}_{r_{t,i}}(t))$ in V'_t are nondegenerate for all $t \in S$ and $i = 1..d_t$

(c) The following conditions are satisfied:

(c1) $h^0(V') = r$

(c2) The set S is finite. Let $d_t = \dim_{\mathbb{C}} K_t(V')$, $\forall t \in S$

(c3) There exists a basis $(s^{(i)}_{t,j})_{t \in S, i=1..d_t, j=1..r_{t,i}}$ of $H^0(V')$ ($\sum_{t \in S, i=1..d_t} r_{t,i} = r$) with the properties :

(c31) $\deg(\Lambda_{i=1..d_t, t' \in S} s^{(i)}_{t',1} \wedge ... \wedge s^{(i)}_{t', r_{t',i}})(t) = \sum_{i=1..d_t} r_{t,i}$, $\forall t \in S$

(c32) $(s^{(i)}_{t,j})_{j=2...r_{t,i}}$ are linearly independent for all $t \in S$ and all $i = 1..d_t$.

(c33) $\deg(s^{(i)}_{t,1} \wedge ... \wedge s^{(i)}_{t,j})(t) = j$, $\forall t \in S$, $\forall i = 1..d_t$, $\forall j = 1..r_{t,i}$

In this case, we have:

$V' \approx \oplus_{t \in S} \oplus_{i=1...d_t} O(t) \otimes F_{r_{t,i}}$

The proof should be rather obvious by now. Instead of writing down all of its details, let us try to make the statement of the theorem look less formidable. Clearly the bundles $W^{(i)}_{r_{t,i}}(t)$ are isomorphic to $O(t) \otimes F_{r_{t,i}}$, while $W^{(i)}_j(t) \approx O(t) \otimes F_j$ give their canonical filtrations. d_t is the number of different indecomposable bundles which multiply $O(t)$ in the decomposition of V'. These bundles are just $W^{(i)}_{r_{t,i}}(t)$, and have ranks $r_{t,i}$ (of which some may coincide). Conditions $(b32)$ and $(b33)$ or, equivalently, conditions $(c32)$ and $(c33)$ are needed to assure that $W^{(i)}_{r_{t,i}}(t) \approx F_{r_{t,i}}$. Conditions

$(b31)$, respectively $(c31)$ are needed in order to have a *direct* factor of the form $O(t) \otimes \oplus_{i=1..d_t} F_{r_{t,i}}$ in the decomposition of V'.

Note that the spectral divisor is:

$$\Sigma_V = \sum_{t \in S} \sum_{i=1..d_t} r_{t,i}\ t \tag{40}$$

We immediately obtain [7]:

Corrolary 2.1 *Let V be a degree zero semistable holomorphic vector bundle over E and $V' = V \otimes O(p)$. Let $s_1 ... s_r$ be a $\mathbb{C}$-basis of $H^0(V')$. Then the spectral divisor of V is given by :*

$$\Sigma_V = (s_1 \wedge ... \wedge s_r) \tag{41}$$

Proof: Since $(s_1 \wedge ... \wedge s_r)$ is independent of the choice of the basis of sections $s_1 ... s_r$, we can choose $s_1 .. s_r$ to have the properties listed in (c) of Theorem 2.2. Then the conclusion is obvious. $\Box$

This shows that the spectral divisor can be computed by an obvious adaptation of the methods of [1] even in the general case. However, the divisor $(s_1 \wedge ... \wedge s_r)$ alone cannot give us enough information to test semistability and/or determine the splitting type.

Starting from the above theorem, it is relatively straightforward to develop an algorithm for testing semistability of V and determining its splitting type by doing a series of simple manipulations on an arbitrary basis of $H^0(V')$. Instead of presenting the algorithm in its full generality (which requires introducing a slightly tedious amount of notation), we will show explicitly how this can be implemented in the simpler case when one is interested in identifying degree zero *fully decomposable* semistable bundles. This is explained in section 3 below.

2.4 The spectral divisor in the monad case and a 'moduli problem'

In this subsection we consider the case when V is given by the cohomology of a monad:

$$0 \longrightarrow \oplus_{j=1..s} O_E \xrightarrow{f} \oplus_{a=1..m} O(D_a) \xrightarrow{g} O(D_0) \longrightarrow 0 \tag{42}$$

Here D_a, D_0 are some divisors on E. We define the twisted bundles and exact sequences as before. We denote all twisted objects by a prime. As usual, we twist by $O(p)$ with p an arbitrary point on $E.p$ is fixed throughout the following discussion. We have $m = r + s + 1$ where $r := \mathrm{rank} V$.

Write (42) as the pair of exact sequences :

$$0 \longrightarrow \ker g \hookrightarrow \oplus_{a=1..m} O(D_a) \xrightarrow{g} O(D_0) \longrightarrow 0 \tag{43}$$

[7] This result can also be obtained without making use of Theorem 2.2

$$0 \longrightarrow \oplus_{j=1..s} O_E \xrightarrow{f} \ker g \xrightarrow{p} V \longrightarrow 0 \tag{44}$$

By taking degrees we obtain :

$$\deg V = \sum_{a=1..m} \deg D_a - \deg D_0 = \deg(\ker g) \tag{45}$$

We have:

Proposition 2.8 *The following are equivalent :*
(a) V is semistable and of degree zero
(b) $\ker g$ *is semistable and of degree zero*

Proof:
Assume that (a) holds. Then the sequence (44) shows that $\ker g$ is an extension of $\oplus_{j=1..s} O_E$ by V. As both these bundles are semistable and of slope zero, a standard result of Seshadri (see. for example, [15]) immediately entails (b).

Assume (b) holds. Then (44) shows that $V = \mathrm{coker} f$ and since $\oplus_{j=1..s} O_E$ and $\ker g$ are both semistable and of slope zero we can use another result of Seshadri to obtain (a). $\square$

This proposition reduces the study of semistabilty of V to that of $\ker g$. In particular, we see that semistability of V depends only on the properties of the map g and on the bundles $\oplus_{a=1..m} O(D_a)$ and $O(D_0)$.

For the following we assume that $\oplus_{a=1..m} \deg D_a = \deg D_0 := d$. with $d \geq 0$. We let $d_a := \deg D_a$. Then (45) assures us that $\deg V = \deg\ker g = 0$.

Now *suppose that V is semistable .*

Then by Proposition 2.8 $\ker g$ is also semistable . Then Lemma 1.1 assures us that $H^1(\ker g') = 0$. Noting that $H^1(O(p))$ also vanishes by the Riemann-Roch theorem, it follows that by twisting the two exact sequences above and taking cohomology we obtain two *short* exact sequences :

$$0 \longrightarrow H^0(\ker g') \hookrightarrow \oplus_{a=1..m} H^0(O(D'_a)) \xrightarrow{g_*} H^0(O(D'_0)) \longrightarrow 0 \tag{46}$$

$$0 \longrightarrow \oplus_{j=1..s} H^0(O(p)) \xrightarrow{f_*} H^0(\ker g') \xrightarrow{p_*} H^0(V') \longrightarrow 0 \tag{47}$$

where $D'_a := D_a + p, D'_0 := D_0 + p$ and we denoted $f \otimes id, g \otimes id$ by the same letters for simplicity. The collapse of the cohomology sequence associated to (43) is a direct consequence of the semistability of $\ker g$.

Since $d+1$ is positive, the Riemann-Roch theorem tells us that $h^0(O(D'_0)) = \deg(D'_0) = \deg D_0 + 1 = d+1$. Since $\ker g$ is semistable and of degree zero, Lemma 1.1 gives $h^0(\ker g') = \mathrm{rank}(\ker g) = r+s = m-1$; then (46) gives $h^0(\oplus_{a=1..m} O(D'_a)) = m + d$. This last fact is not a consequence of Riemann-Roch unless d_a are all nonnegative.

Proposition 2.9 *Let $\Sigma_{\ker g}$ and Σ_V be the spectral divisors of* $\ker g$*, respectively V. Then $\Sigma_{\ker g} = \Sigma_V + sp$.*

Proof: Since (47) is an exact sequence of vector spaces, it must split. We can thus choose a basis $v_1...v_{r+s}$ of $\ker g'$ with the properties :

(1)$v_1 = f_*(w_1)...v_s = f_*(w_s)$, where $w_1...w_s$ is a basis of $A := \oplus_{j=1..s} H^0(O(p))$

(2)$p_*(v_{s+1}) := u_1...p_*(v_{s+r}) := u_r$ is a basis of $H^0(V')$

The canonical isomorphism $\det(\ker g') \approx \det(A) \otimes \det(V')$ maps the section $v_1 \wedge .. \wedge v_{r+s} \in H^0(\det(\ker g'))$ into the the section $(w_1 \wedge ... \wedge w_s) \otimes (u_1 \wedge ... \wedge u_r) \in H^0(\det A) \otimes H^0(\det V') \subset H^0(\det A \otimes \det V')$. Thus:

$$\Sigma_{\ker g} = (v_1 \wedge ... \wedge v_{r+s}) = (w_1 \wedge ... \wedge w_s \otimes u_1 \wedge ... \wedge u_r) = (w_1 \wedge ... \wedge w_s) + (u_1 \wedge ... \wedge u_r) = sp + \Sigma_V \tag{48}$$

where in the first and last line we used the corrolary to Theorem 2.2. □

The relation between $\Sigma_{\ker g}$ and the bundle $B := \oplus_{a=1..m} O(D'_a)$ is more complicated. The reason is that there is no simple connection between the local behaviour of the sections of $\ker g$ and the sections of B[8]. To extract more information about $\ker g$, one has to undertake a more detailed study based on the properties of the map g. In particular, one would like to find necessary and sufficient conditions on g such that $\ker g$ is semistable and describe the associated moduli space of g. Although we will not attempt this here, let us formulate the geometric set-up of the problem.

Theorem 2.2 reduces the semistability condition for $\ker g$ to conditions on the subspace $W := H^0(\ker g')$ of $U := H^0(B)$. We have $\dim_{\mathbb{C}} U = m + d$ and semistability requires that $\dim_{\mathbb{C}} W = m-1$. Let $H_e := \ker \phi_e^B$, for all $e \in E$. Note that $\phi_e^{\ker g'} = \phi_e^B|_W$, so that $\ker \phi_e^{\ker g'} = H_e \cap W$.

Suppose for simplicity that all D_a are effective and that $\mathrm{Card}\{a \in \{1..m\} | d_a = 0\} = \nu$. Since $\phi_e^B = \oplus_{a=1..m} \phi_e^{O(D'_a)}$ for all $e \in E$, we have $\ker(\phi_e^B) = \oplus_{a=1..m} \ker \phi_e^{O(D'_a)}$. With our assumptions, we have $\mathrm{codim}_{\mathbb{C}}(\ker \phi_e^{O(D'_a)}) = 1$ for all a with $d_a > 0$ and all $e \in E$, while for all a with $d_a = 0$ (i.e. $D_a = 0$, $O(D'_a) = O(p)$) we have $\mathrm{codim}_{\mathbb{C}}(\ker \phi_e^{O(D'_a)}) = 1$ for $e \neq p$ and $\mathrm{codim}_{\mathbb{C}}(\ker \phi_p^{O(D'_a)}) = 0$. Thus $\mathrm{codim}_{\mathbb{C}} H_e = m$ for all $e \neq p$ while $\mathrm{codim}_{\mathbb{C}} H_p = m - \nu$.

Note that $\dim_C W + \dim_{\mathbb{C}} H_e = \dim_{\mathbb{C}} U - 1$ for $e \neq p$ while $\dim_{\mathbb{C}} W + \dim \mathbb{C} H_p = \dim_{\mathbb{C}} U + \nu - 1$. For given divisors D_a and a given map g, $W \cap H_e$ will have fixed dimension D for almost all points $e \in E$. The points where the dimension of this intersection increases correspond to the points of the set $\mathcal{Z}(\ker g')$.

If $\nu > 1$, it follows that $\dim_{\mathbb{C}} W \cap H_p \geq \nu - 1$. On the other hand, we cannot deduce any simple lower bound on $\dim_{\mathbb{C}} W \cap H_e$ for $e \neq p$.

Geometrically, we are given a map $H : E \to Sbsp(U)$, $H(e) := H_e, \forall e \in E$ from E to the set of subspaces of the $m + d$-dimensional $\mathbb{C}$-vector space U. The precise form of this map is completely fixed by the bundle B. As e varies in E, H_e describes a complicated trajectory in $Sbsp(U)$. Generically on E, H_e has codimension m, except at the point $e = p$ where it has codimension $m - \nu$.

[8]This happens because typically we have $d_a > 0$ for some a. Then $O(D'_a)$ is quasi-ample (the evaluation map is surjective everywhere), which to complications.

Giving a semistable subbundle of B of the form ker$g$ requires giving the $m-1$ dimensional subspace W of U, with the property that it is complementary to H_e for a generic $e \in E$ and satisfying the other conditions in Theorem 2.2. The precise position of W inside U is controlled by the map g. It is not hard to see that the remaining conditions in the theorem can be expressed in terms of 'incidence relations' constraining the 'speeed of incidence' of W on H_e as $e \to t_i$; this is similar to the discussion of Section 2.

The set-up above allows us to reduce the problem of determining the maps g giving a semistable ker$g$ to a problem in linear algebra and analysis. In particular, it is ideal for extracting information about 'moduli'. The 'trajectory' of H_e is, however, rather complicated in general and the problem may be quite difficult in practice. It would be interesting to investigate this further.

3 The fully split case

3.1 Twist by $O(p)$

Let (E,p) be an elliptic curve with a marked point and V a holomorphic bundle of degree zero and rank r on E. Let $V' := V \otimes O(p)$.

We say that V is *fully split* if there exists a decomposition $V = \oplus_{j=1..r} L_j$ of V into a direct sum of line bundles L_j.

We present an algorithm for determining whether a given degree zero holomorphic vector bundle V is semistable *and* fully split. The algorithm requires explicit knowledge of $H^0(V')$ and allows for the determination of the line bundles L_j up to holomorphic equivalence.

Theorem 3.1 *Let V be a degree zero holomorphic vector bundle of rank r over E. Let $R_e := R_e(V')$, $r_e := \dim_{\mathbb{C}} R_e$, $K_e := K_e(V')$ and $d_e := \dim_{\mathbb{C}} K_e$ for any $e \in E$.*

The following statements are equivalent :

(a) V is semistable and fully split

(b) V' satisfies all of the following conditions :

(b0) $h^0(V') = r$

(b1) The set $S := \mathcal{Z}(V') = \{t \in E | r_t < r\}$ is finite

(b2) For all $t \in S$, all holomorphic sections of V' belonging to $K_t - 0$ have degree 1 at t

(b3) For each $t \in S$ we have $V'_t = R_t \oplus N_t$

(b4) We have $H^0(V') = \oplus_{t \in S} K_t$

(c) V' satisfies (b0),(b1), (b4) and the condition that there exits a basis $(s_1..s_r)$ of $H^0(V')$ such that :

(b23) $\mathrm{deg} s_1 \wedge s_2 \wedge ... \wedge s_r(t) = d_t, \forall t \in S$

Moreover, in this case we have $V \approx \oplus_{t \in S} O(t-p)^{\oplus d_t}$.

Note that if $(b23)$ holds for a basis of $H^0(V')$ then it will hold for any other basis.

Proof:

Show that (a) implies (b):

Assume (a) holds and write $V = \oplus_{i=1..r} L_i$ with $L_i \in \mathrm{Pic}^0(E)$. Then $L_i \approx O(q_i - p)$ $(q_i \in E)$ and $V' = \oplus_{i=1..r} L'_i$, with $L'_i = L_i \otimes O(p) \approx O(q_i)$.

We know that $(b0)$ holds by Lemma 1.1. Let $s_i \in H^0(L'_i) - \{0\}$. Then $s_1 ... s_r$ is a $\mathbb{C}$-basis of $H^0(V')$. We obviously have $S = \cup_{i=1..r}\{q_i\}$, so $(b1)$ holds. Since V' is semistable of slope 1 we see that $(b2)$ also holds (cf. the remark before Proposition 2.1).

Now suppose there are two distinct points $t_1, t_2 \in S$ such that $K_{t_1} \cap K_{t_2} \neq \{0\}$. Let $s \in K_{t_1} \cap K_{t_2}$. Then $s(t_1) = s(t_2) = 0$ and, since s is regular, we must have $\deg L_s \geq 2$, which contradicts semistability of V'. Thus the sum $\sum_{t \in S} K_t$ is direct. On the other hand, any $s \in H^0(V')$ is a linear combination $s = \sum_{i=1..r} \alpha_i s_i$ $(\alpha_i \in \mathbb{C})$. Since $s_i \in K_{q_i}$, we have $s \in \sum_{t\in S} K_t$. Thus $(b4)$ holds.

To show $(b3)$, note that $L'_i = L_{s_i}$ (in the notation of subsection 2.1). Fixing $t \in S$, we clearly have $R_t = \oplus_{i; q_i \neq t} (L'_i)_t$, $K_t = \oplus_{i;q_i = t} H^0(L'_i)$ and $N_t = \oplus_{i;q_i=t}(L'_i)_t$. Since $V'_t = \oplus_{i=1..r}(L'_i)_t$, we have $V'_t = R_t \oplus N_t$.

Show that (b) implies (a):

Let $d_t := \dim_{\mathbb{C}} K_t$ $(t \in S)$. By $(b4)$, we can choose a $\mathbb{C}$-basis $(s_j^{(t)})_{t\in S, j=1..d_t}$ of $H^0(V')$ such that $(s_j^{(t)})_{j=1..d_t}$ is a $\mathbb{C}$ -basis of K_t for each $t \in S$. By $(b4)$ and $(b2)$, each section $s_i^{(t)}$ has exactly one zero on E, namely at t, and this zero is simple (the unicity of this zero easily follows from $(b4)$). Therefore the line bundles $L_j^{(t)} := L_{s_j^{(t)}}$ have degree 1 and we have $L_j^{(t)} \approx O(t)$. In particular, for all $j = 1..d_t$ we have $s_j^{(t)}(t) = 0$ and $\hat{s}_j^{(t)}(t) \neq 0, \forall j = 1..d_t$ and $s_j^{(t)}(t') \neq 0, \forall t' \in S - \{t\}$. Moreover, $(b3)$ implies that $s_j^{(t')}(t)(t' \in S - \{t\}, j = 1..d_{t'})$ and $\hat{s}_j^{(t)}(t)(j = 1..d_t)$ form a basis of V'_t. Therefore, we have $V'_t = \oplus_{t' \in S, j=1..d_{t'}} (L_j^{(t')})_t$, $\forall t \in S$. On the other hand, for all $e \in E - S$ we have $\dim_{\mathbb{C}} R_e = r$. Since $(s_j^{(t)}(e))_{t\in S, j=1..d_t}$ obviously generate R_e and since $(b4)$ implies that $\mathrm{Card}\{s_j^{(t)} | t \in S, j = 1..d_t\} = r$, it must be the case that $(s_j^t(e))_{t \in S, j=1..d_t}$ is a $\mathbb{C}$-basis of V'_e, for all $e \in S - t$. Therefore, we also have $V'_e = \oplus_{t\in S, i=1..d_t} (L_i^{(t)})_e$, for $e \in E - S$. Therefore, $V' = \oplus_{t \in S, j=1..d_t} L_j^{(t)}$. Since each component of this sum has slope 1, it follows that V' is semistable and of slope 1, while $V = V' \otimes O(-p)$ is semistable and of degree zero. We also have $V' \approx \oplus_{t\in S} O(t)^{d_t}$ and $V \approx \oplus_{t \in S} O(t-p)^{d_t}$.

Show that (b) and (c) are equivalent

For this, assume that $(b0)$, $(b1)$ and $(b4)$ hold. Then we show that $(b2)$ and $(b3)$ together are equivalent to $(b23)$.

Remember that $\deg s_1 \wedge ... \wedge s_r(e)$ does not depend on the choice of the $\mathbb{C}$-basis of $H^0(V')$. Enumerating $S = \{t_1..t_k\}$ we can assume that $(s_i)_{d_1+...+d_{j-1}+1 \leq i \leq d_1 + ... + d_j}$ is a $\mathbb{C}$-basis of K_{t_j} for all $j = 1..k$. Since the argument is similar for each j, let us focus on $t_1 := t$. Then $s_1 ... s_{d_t}$ is a basis of K_t and for $i = 1..d_t$ we have $s_i(e) = z\sigma_i(e)$ for all

e close to t, where σ_i are local holomorphic sections of V' around t.

Claim 1: If $(b0)$, $(b1)$ and $(b4)$ hold then $s_{d_t+1}(t)...s_r(t)$ is a basis of R_t.

Indeed, since $s_1(t) = ...s_{d_t}(t) = 0$, we clearly have that $s_{d_t+1}(t)...s_r(t)$ generate R_t. If $\alpha_{d_t+1}s_{d_t+1}(t) + ... + \alpha_r s_r(t) = 0$ is zero a linear combination, then the section $s := \alpha_{d_t+1}s_{d_t+1} + ... + \alpha_r s_r$ of V' vanishes at t so that it belongs to K_t. Since our basis $s_1...s_r$ is 'adapted' to the decomposition $(b4)$, s then gives an element of $K_t \cap (\sum_{t' \in S-\{t\}} K_{t'})$, which must be zero since the sum in $(b4)$ is direct. Since $s_{d_t+1}...s_r$ are $\mathbb{C}$-linearly independent, this implies that $\alpha_{d_t+1} = .. = \alpha_r = 0$. Thus $s_{d_t+1}(t)...s_r(t)$ are linearly independent and the claim is proven.

Claim 2: If $(b0)$, $(b1)$ and $(b4)$ hold then the following are equivalent:

(α) $(b2)$ holds at t

(β) $\sigma_1(t)...\sigma_{d_t}(t)$ are linearly independent

In this case, $\sigma_1(t)...\sigma_{d_t}(t)$ form a basis of N_t.

To prove this, first assume that $(b2)$ holds at t. Consider a zero linear combination $\alpha_1\sigma_1(t) + ... + \alpha_{d_t}\sigma_{d_t}(t) = 0$. If the section $s := \alpha_1 s_1(t) + ... + \alpha_{d_t}s_{d_t}$ would be nonzero, then it would have vanishing degree at least 2 at t. This would contradict $(b2)$. Therefore, we must have $s = 0$ and $\alpha_1 = ..\alpha_{d_t} = 0$. This proves that (α) implies (β).

Now assume that (β) holds and consider a section $s \in K_t - \{0\}$. Then $s = \alpha_1 s_1(t) + ... + \alpha_{d_t}s_{d_t}$ for some $\alpha_i \in \mathbb{C}$ so that $s(e) = z(\alpha_1\sigma_1(e) + ... + \alpha_{d_t}\sigma_{d_t}(e))$ for e close to t. Since s is not the zero section, at least one α_i is nonzero and (β) implies that $\alpha_1\sigma_1(t) + ... + \alpha_{d_t}\sigma_{d_t}(t)$ is nonzero. Thus s has degree 1 at t and (α) holds.

Assume that the equivalent conditions (α), (β) hold and show that $\sigma_1(t)...\sigma_{d_t}(t)$ generate N_t. We have $N_t :=< A >$, where $A := \{\hat{s}(t) | s \in K_t\}$. If $s \in K_t - \{0\}$, the above arguments show that $\hat{s}(t)$ belongs to $< \sigma_1(t)...\sigma_{d_t}(t) >$, and this is also trivially true for $s = 0$ (since $\hat{s}(t) = 0$ by definition in this case). Therefore we have $A \subset< \sigma_1(t)...\sigma_{d_t}(t) >$ and $\sigma_1(t)...\sigma_{d_t}(t)$ generate N_t. This finishes the proof of Claim 2.

Now return to the proof of the theorem. Since $s_1(e) \wedge ... \wedge s_r(e) = z(\sigma_1(e) \wedge ... \wedge \sigma_{d_t}(e) \wedge s_{d_t+1}(e) \wedge ... \wedge s_r(e))$ for e close to t, $(b23)$ is equivelent to the statement that $\sigma_1(t)...\sigma_{d_t}(t), s_{d_t+1}(t)...s_r(t)$ is a basis of V'. By Claim 1, linear independence of $s_{d_t+1}(t)...s_r(t)$ is automatic and $< s_{d_t+1}(t)...s_r(t) >= R_t$. By Claim 2, linear independence of $\sigma_1(t)...\sigma_{d_t}(t)$ is equivalent to $(b2)$ and in this case $< \sigma_1(t)...\sigma_{d_t}(t) >= N_t$. Then $< \sigma_1(t)...\sigma_{d_t}(t), s_{d_t+1}(t)...s_r(t) >= V'_t$ is equivalent to $(b3)$. □

Let us explain how one can test $(b4)$. Suppose that $(b0), (b1)$ hold and let $s_1...s_r$ be an arbitrary $\mathbb{C}$-basis of $H^0(V')$. For each $t \in S$, consider the d_t -dimensional subspace P_t of $\mathbb{C}^r$ of linear relations among $s_1(t)...s_r(t)$:

$P_t := \{a := (a_1...a_r) \in \mathbb{C}^r | a_1 s_1(t) + .. + a_r s_r(t) = 0\}$

Choose vectors $a^{(t,j)} \in \mathbb{C}^r$ $(t \in S, j = 1..d_t)$ such that, for each $t \in S$ $(a^{(t,j)})_{j=1..d_t}$ is a basis of P_t. Let $\zeta^{(t,j)} := \sum_{i=1..r} a_i^{(t,j)}\ s_i \in H^0(V')$. Then $(\zeta^{(t,j)})_{j=1..d_t}$ is a basis of K_t for all $t \in S$. In particular, we have $d_t = \dim_{\mathbb{C}} P_t$. Clearly $(b4)$ is equivalent to the

condition:

$$\mathbb{C}^r = \oplus_{t\in S} P_t \tag{49}$$

Chosing an enumeration $S = \{t_i | i = 1..k\}$ of S, we can form a matrix $A \in Mat(d, r, C)$, whose lines are given by the vectors $(a^{(t_i,j)})_{i=1..k, j=1..d_t}$. Then $(b4)$ is equivalent to the conditions $d = r$ and $\det A \neq 0$. Therefore, we obtain the following

Algorithm :

Suppose V is a rank r and degree zero holomorphic vector bundle over E. Let $p \in E$ arbitrary and define $V' := V \otimes O(p)$.

Step 1:
Obtain a basis $(s_1 ... s_n)$ of $H^0(V')$.

Step 2:
If $n \neq r$ then V is not semistable. Otherwise, continue with Step 3.

Step 3:
Let $\delta := s_1 \wedge ... \wedge s_r \in H^0(\Lambda^r V')$. If $\delta = 0$ then V is not semistable (this follows from the main theorem in section 2). Otherwise, the set $S := \mathrm{supp}(\delta)$ is finite. In this case, enumerate $S = \{t_1..t_k\}$ and continue with Step 4.

Step 4 :
For each $t \in S$, determine $d_t = \dim_{\mathbb{C}} K_t$[9]. Then V' is semistable and fully split iff each of the following conditions is satisfied:
(a) $\sum_{t\in S} d_t = r$
(b) $\deg s_1 \wedge ... \wedge s_r(t) = d_t$ for all $t \in S$
(c) The matrix A is nonsingular
In this case, we have $V' \approx \oplus_{t\in S} O(t)^{\oplus d_t}$. In particular, the spectral divisor of V is given by :

$$\Sigma_V = \sum_{t\in S} d_t\ t = (s_1 \wedge ... \wedge s_r) \tag{50}$$

Note that $\mathrm{supp}\Sigma_V = S$.

3.2 More general twists

Let V be a fully split semistable vector bundle of degree zero over E. Then $V = \oplus_{j=1..r} L_j$ with $L_j \in \mathrm{Pic}^0(E)$.

[9] In general we can determine d_t as $d_t = \dim_{\mathbb{C}} P_t$. In the monad case, V' has a natural embedding into a direct sum of line bundles and d_t can be determined directly by considering the rank of a matrix of sections as in [1]

Let $D = p_1 + ... + p_h$ be an effective divisor on E, where $p_1 ... p_h$ are *mutually distinct* points on E. We use p_1 as a base point of E. Then we can write $L_j \approx O(q_j - p_1)$ with $q_j \in E$. Define:

$$V' := V \otimes O(D) = \oplus_{j=1..r} L'_j \tag{51}$$

where $L'_j := L_j \otimes O(D) \approx O(q_j + p_2 + ... + p_h)$.

Since $\deg L'_j = h$, we have $h^0(L'_j) = h$ and $h^0(V') = rh$. The Riemann-Roch theorem gives $h^1(V') = 0$. The spectral divisor of V is $\Sigma_V = \sum_{j=1..r} q_j$. Let $S := \mathrm{supp}\Sigma_V$. For each $q \in S$, let $S_q = \{j \in \{1..r\} | q_j = q\}$ and $d_q := \mathrm{Card} S_q$. Then $L'_j \approx O(q + p_2 + .. + p_h)$ for all $j \in S_q$.

Lemma 3.1 *Let $q \in E$ be arbitrary. The set :*

$$G_q(D) := \{s \in H^0(O(q + p_2 + ... + p_h)) | s(p_j) = 0, \forall j = 2..h\} \tag{52}$$

is a one-dimensional subspace of the $\mathbb{C}$-vector space $H^0(O(q+p_2+...+p_h))$. Moreover, for any $s \in S_q(D) - \{0\}$ we have :

$$(s) = q + p_2 + p_3 + ... + p_h \tag{53}$$

Proof: Obviously the zero section belongs to $G_q(D)$. Now let $s \in G_q(D) - \{0\}$. Since $s(p_2) = ... = s(p_h) = 0$, we have :

$$(s) = D_s + p_2 + ... + p_h \tag{54}$$

with D_s an effective divisor. Since $s \in H^0(O(q + p_2 + ... + p_h))$, we have $\deg(s) = \deg(q + p_2 + ... + p_h) = h$. But $\deg(s) = \deg D_s + (h-1)$ by (54). Thus $\deg D_s$=1. Since D_s is effective this implies $D_s = q'$ for some $q' \in E$. On the other hand, $s \in H^0(O(q + p_2 + ... + p_h))$ implies $(s) \sim q + p_2 + ... + p_h$, where $\sim$ denotes linear equivalence. Together with (54), this gives $q' \sim q$. If $q' \neq q$, this would imply $E \approx \mathbb{P}^1$ by a classical theorem. Thus we must have $q' = q$ and $(s) = q + p_2 + ... + p_h$ for all $(s) \in G_q(D) - \{0\}$. By a standard argument this implies that any $s' \in G_q(D) - \{0\}$ is of the form $s' = \lambda s$ with $\lambda \in \mathbb{C}^*$ a constant. Thus $G_q(D)$ is a one dimensional $\mathbb{C}$-vector space. $\Box$

Let :

$$G_j := \{s \in H^0(L'_j) | s(p_2) = ... = s(p_h) = 0\} \approx G_{q_j}(D) \tag{55}$$

$(j = 1..r)$. By Lemma 3.1, G_j are one-dimensional subspaces of $H^0(V')$. Define :

$$G := \{s \in H^0(V') | s(p_2) = ... = s(p_h) = 0\} \subset H^0(V') \tag{56}$$

and:

$$G(q) = \oplus_{j \in S_q} G_j \subset G \tag{57}$$

(for all $q \in S$).

Proposition 3.1 *We have :*

$$G = \oplus_{j=1..r} G_j = \oplus_{q \in S} G(q) \tag{58}$$

In particular, G is an r-dimensional subspace of the rh dimensional $\mathbb{C}$-vector space $H^0(V')$. Moreover, for any $s \in G - \{0\}$ we have the alternative :

Either

(a) $(s) = q + p_2 + ... + p_h$ for some $q \in S$

or

(b) $(s) = p_2 + ... + p_h$

If (a) holds then $s \in G(q)$ for some $q \in S$, while if (b) holds then $s \in G - \cup_{q \in S} G(q)$.

Here the equalities are between divisors and *not* between divisor classes. That is, the equality in (a) and (b) is to be taken at face value and *not* in the sense of linear equivalence.

Proof:

Since $V' = \oplus_{j=1..r} L'_j$, the statement $G = \oplus_{j=1..r} G_j$ is obvious.

Now let $s \in G - \{0\}$. By Proposition 2.3, we have:

$$\deg(s) \leq \mu(V') = h \tag{59}$$

Since (s) is effective and $p_2 ... p_r \in \text{supp}(s)$, there are only two possibilities :

(a) $(s) = q + p_2 + ... + p_h$ for some $q \in E$ (note that q may belong to the set $\{p_2 ... p_h\}$), and in this case $\deg(s) = h$

(b) $(s) = p_2 + ... + p_h$, and in this case $\deg(s) = h - 1$.

Using $G = \oplus_{j=1..r} G_j$, we can write:

$$s = \oplus_{j=1..r} s_j \tag{60}$$

where $s_j \in G_j$. From $s_j \in G_j \subset H^0(L'_j)$, we obtain $(s_j) = q_j + p_2 + ... + p_h$ unless $s_j = 0$. Since (60) is a direct sum, we have :

$$(s_j) \geq (s) \text{ unless } s_j = 0 \tag{61}$$

for all $j = 1..r$ [10]. Indeed, s can have a zero of order m at $e \in E$ iff each s_j has a zero of order at least m at e.

In case (a), (61) shows that $(s_j) = (s)$ for all $j = 1..r$ with s_j different from zero. This set is nonvoid iff $q \in S$ and in this case we obtain $s = \sum_{j \in S_q} s_j \in G(q)$.

In case (b), we cannot have $s \in G(q)$ for any q, since obviously this would imply $(s) \geq q + p_2 + ... + p_h$, a contradiction. □

Definition 3.1 *A $\mathbb{C}$-basis $\sigma_1 ... \sigma_r$ of G is called* canonical *if* $\deg \sigma_j = h$, *for each* $j = 1..r$.

[10]This means that $\deg(s_j)(e) \geq \deg(s)(e)$ for all $e \in E$.

By the previous proposition, a basis of G is canonical iff it is adapted to the graduation (58) of G, i.e. iff it is of the form $(\sigma_j^q)_{q\in S, j=1..d_q}$ with $(\sigma_j^q)_{j=1..d_q}$ bases of $G(q)$.

Corrolary 3.1 *Let $s_1...s_r$ be an arbitrary basis of G. Then the spectral divisor of V is given by :*

$$\Sigma_V = (s_1 \wedge .... \wedge s_r) - r(p_2 + ... + p_d) \tag{62}$$

Proof: Indeed, if $\sigma_1...\sigma_r$ is a canonical basis of G then we have $(s_1 \wedge ... \wedge s_r) = (\sigma_1 \wedge ... \wedge \sigma_r) = \sum_{j=1..r} q_j + r(p_2 + ... + p_r) = \Sigma_V + r(p_2 + ... + p_r)$. $\square$

This reduces the problem of determining the spectral divisor of V to finding a basis of G. In the monad case, that can be easily accomplished by an obvious modification of the methods of [1].

It is now straightforward to formulate an analogue of Theorem 3.1, in which $H^0(V')$ is replaced by G, whose proof is almost identical. Since this brings no new concepts to bear, we will not insist.

There is a also a relatively straightforward generalization of the above to the non-fully-split case. A detailed statement would be rather lengthy and will not be given here.

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